

A CRITERIA OF STRONG H-DIFFERENTIABILITY

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Abstract: We give a criteria for a Malliavin differentiable function to be strongly H-differentiable.

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1. Introduction

Let \mathbb{W} be the classical Wiener space and H the associated Cameron-Martin space. A theory of a weak derivative over Wiener functional with respect to H directions has long been developed (see [3], [5]). More recently, Üstünel and Zakai, in [2], or Kusuoka, in [1] have studied a strong derivative for Wiener functional, using the Fréchet differentiability on H . A Wiener functional f is H -continuous, or H -C, if $h \mapsto f(w+h)$ is a.s. continuous on H , $H-C^1$ if $h \mapsto f(w+h)$ is a.s. Fréchet differentiable on H with H -continuous derivative. $H-C^1$ function are very useful in the study of invertibility of perturbations of the identity of Wiener space. Indeed if u is a measurable $H-C^1$ function from \mathbb{W} to H , $I_{\mathbb{W}} + u$ is invertible. This has been used by Üstünel to establish the following variational representation, where B is a Brownian motion

$$-\log \mathbb{E} [e^{-f \circ B}] = \inf_u \mathbb{E} \left[f \circ (B + u) + \frac{1}{2} \int_0^1 |\dot{u}(s)|^2 ds \right]$$

for some unbounded functions f .

Of course it is way more difficult to establish that a function is $H-C^1$ than it is to establish it is weakly H -differentiable. In this paper, we give a criteria for a weakly H -differentiable function to be $H-C^1$, namely the weak H -derivative has to be a.s. uniformly continuous on every zero-centered ball of H .

First we recall the formal setting of weak and strong H -derivative, then we establish the criteria. Finally, we expand the criteria to higher order derivatives.

2. Framework

Set $n \in \mathbb{N}$ and let \mathbb{W} be the canonical Wiener space $C([0, 1], \mathbb{R}^n)$. Let H be the associated Cameron-Martin space

$$H = \left\{ \int_0^\cdot \dot{h}(s) ds, \dot{h} \in L^2([0, 1]) \right\}$$

and for $m \in \mathbb{N}^*$, $B_m = \{h \in H, |h|_H \leq m\}$. Denote μ the Wiener measure and W the coordinate process. W is a Brownian motion under μ and we denote (\mathcal{F}_t) the canonical filtration of W completed with respect to μ . Set Cyl the set of cylindrical functions

$$Cyl = \{F(W_{t_1}, \dots, W_{t_p}), p \in \mathbb{N}^*, F \in \mathcal{S}(\mathbb{R}^n), 0 \leq t_1 < \dots < t_p \leq 1\}$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the set of Schwartz functions on \mathbb{R}^n .

For $f \in Cyl$, $w \in W$ and $h \in H$, we define

$$\nabla_h f(w) = \left. \frac{d}{d\lambda} f(w + \lambda h) \right|_{\lambda=0}$$

Riesz theorem enables us to consider ∇f as an element of H . For $1 < p < \infty$, we define

$$|\cdot|_{p,1} : f \in Cyl \mapsto |f|_{L^p(\mu)} + |\nabla f|_{L^p(\nu, H)}$$

∇f is a closable operator and we define $\mathbb{D}_{p,1}$ the closure of Cyl for $|\cdot|_{p,1}$.

Let δ be the adjoint operator of ∇ and $L_a^0(\mu, H)$ be the set of the element of $L^0(\mu, H)$ whose density are adapted to (\mathcal{F}_t) . $L_a^0(\mu, H)$ is a subset of the domain of δ , and for any $u \in L_a^0(\mu, H)$

$$\delta u = \int_0^1 \dot{u}(s) dW(s)$$

From now on for $u \in L_a^0(\mu, H)$, we will denote

$$\rho(\delta u) = \exp \left(\delta u - \frac{1}{2} \int_0^1 |\dot{u}(s)|^2 ds \right)$$

Now set X a separable Hilbert space and $(e_i)_{i \in \mathbb{N}}$ an Hilbert base of X , define

$$Cyl(X) = \left\{ \sum_{k=1}^p f_i e_{i_k}, p \in \mathbb{N}^*, (i_k) \in \mathbb{N}^p, (f_i) \in Cyl^p \right\}$$

If $f = \sum_{k=1}^p f_i e_{i_k} \in Cyl(X)$, we define

$$\nabla f(w)[h] = \sum_{k=1}^p \nabla_h f_i e_{i_k}$$

and ∇f is an element of $X \otimes H$.

We define $|\cdot|_{p,1}$, similarly as before and ∇ is once again a closable operator, we define $\mathbb{D}_{p,1}(X)$ the closure of $Cyl(X)$ for $|\cdot|_{p,1}$. This enables us to define ∇^p for $p \geq 1$ by recurrence, we denote

$$|\cdot|_{p,k} : f \mapsto |f|_{L^p(\mu, X)} + \sum_{k=1}^p |\nabla^k f|_{L^p(\mu, X \otimes H^{\otimes k})}$$

and $\mathbb{D}_{p,k}(X)$ the completion of $Cyl(X)$ for $|\cdot|_{p,k}$.

Finally, we define the Ornstein-Uhlenbeck semigroup (P_t) as follow: set $t > 0$ and $f \in L^p(\mu, X)$ for some $p \geq 1$

$$P_t f : w \in \mathbb{W} \mapsto \int_{\mathbb{W}} f \left(e^{-t} w + \sqrt{1 - e^{-2t}} y \right) \mu(dy)$$

We will need the following technical results concerning P_t :

Proposition 1. *Set $t > 0$ and $f \in L^1(\mu, X)$. For $h \in H$, we have μ -a.s.*

$$P_t f(w + h) = P_t \left((f \circ (\cdot + e^{-t} h)) \right) (w)$$

If f belongs to some $\mathbb{D}_{p,1}(X)$, we have μ -a.s.

$$P_t \nabla f = e^t \nabla P_t f$$

Proof: For the sake of simplicity we address the case $X = \mathbb{R}$. The first assertion is an easy calculation.

For the second one, set $h \in H$, we have

$$\nabla \rho(\delta h) = h \rho(\delta h)$$

and

$$P_t \rho(\delta h) = \rho(\delta(e^{-t}h))$$

so

$$\begin{aligned} P_t \nabla \rho(\delta h) &= h \rho(\delta(e^{-t}h)) \\ &= e^t e^{-t} h \rho(\delta(e^{-t}h)) \\ &= e^t \nabla \rho(\delta(e^{-t}h)) \\ &= e^t \nabla P_t \rho(\delta h) \end{aligned}$$

and we conclude with density of the vector space generated by $\{\rho(\delta h), h \in H\}$ in $\mathbb{D}_{p,1}$. \square

For more details on this setting see [3] or [5].

Now we give the definitions of strongly H-differentiable functions.

Definition 1. Set $u : \mathbb{W} \rightarrow X$ a measurable function. We say that

- (i) u is H -continuous (or H -C) if the map $h \mapsto u(w+h)$ is μ -a.s. continuous on H .
- (ii) u is H -C¹ if the map $h \mapsto u(w+h)$ is μ -a.s. Fréchet-differentiable and its Fréchet derivative ∇f is an H -continuous map from \mathbb{W} to $X \otimes H$.
- (iii) Set $p \in \mathbb{N}$, by recurrence, u is H -C^p if it is H -C^{p-1} and its derivative of order $p-1$ is an H -C¹ map from \mathbb{W} to $X \otimes H^{\otimes p-1}$.

We will need the following results concerning strong H-regularity for our main theorem, see [2] for their proof:

Proposition 2. Set $u : \mathbb{W} \mapsto X$ such that $\nabla^k u$ is well-defined for every $k \in \mathbb{N}^*$. Assume there exists $p \in \mathbb{N}^*$ such that for every $\lambda \in \mathbb{R}_+$

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |\nabla^k u|_{L^p(\mu, X \otimes H^{\otimes k})} < \infty$$

Then μ -a.s. for every $h \in H$

$$u(w+h) = \sum_{k=0}^{\infty} \frac{1}{k!} \nabla^k u(w) [h^{\otimes k}]$$

Proposition 3. Set $f \in L^p(\mu, X)$ for some $p > 1$, for every $t > 0$ and $\lambda \in \mathbb{R}_+$, we have

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |\nabla^k P_t f|_{L^p(\mu, X \otimes H^{\otimes k})} < \infty$$

3. Main theorem

Theorem 1. *Assume that $f : \mathbb{W} \rightarrow X$ is in $\mathbb{D}_{p,1}(X)$ for some $p > 1$. Assume that $h \mapsto \nabla f(w + h)$ uniformly continuous on every n -ball of H . Then f is $H - C^1$ and its H -derivative is ∇f .*

Proof: The hypothesis implies that $h \mapsto \nabla f(w + h)$ is separable so the uniform continuity hypothesis can be written:

$$\lim_{\epsilon \rightarrow 0} \sup_{h, k \in B_n, |h - k|_H \leq \epsilon} |\nabla f(w + h) - \nabla f(w + k)|_{X \otimes H} = 0 \quad a.s.$$

As we just stated we can set $A \subset \mathbb{W}$ of full measure such that for every $w \in \mathbb{W}$ $h \mapsto \nabla f(w + h)$ is continuous.

Set $s > 0$ and $h \in H$. We know the action of P_s over the weak derivative:

$$P_s \nabla f(w + h) = e^s \nabla P_s f(w + h) \quad a.s.$$

We also have:

$$P_s \nabla f(w + h) = P_s (\nabla f(\cdot + e^{-s} h))(w) \quad a.s.$$

Since both terms are analytic, the set on which these equalities hold does not depend on h . Now we denote, for $m, n \in \mathbb{N}^*$:

$$\theta_{nm}(w) = \sup_{h, k \in B_n, |h - k|_H \leq \frac{1}{m}} |\nabla f(w + h) - \nabla f(w + k)|_{X \otimes H}$$

Observe that for $h, k \in B_n$ verifying $|h - k|_H < \frac{1}{m}$, we have:

$$|P_s (\nabla f(\cdot + e^{-s} h))(w) - P_s (\nabla f(\cdot + e^{-s} k))(w)|_{X \otimes H} \leq P_s \theta_{nm}(w) \quad a.s.$$

Since both terms have analytic modifications, the set of w on which this inequality stands is independent of h and k .

Set (s_i) a sequence decreasing towards 0 and H_0 a countable dense subset of H . We define:

$$\begin{aligned} A' &= A \\ &\cap \{w \in \mathbb{W} : P_{s_i} \nabla f(w + h) = e^{s_i} \nabla P_{s_i} f(w + h), \forall h \in H, \forall i \in \mathbb{N}\} \\ &\cap \{w \in \mathbb{W} : P_{s_i} \nabla f(w + h) = P_{s_i} (\nabla f(\cdot + e^{-s_i} h))(w), \forall h \in H, \forall i \in \mathbb{N}\} \\ &\cap \left\{w \in \mathbb{W} : \lim_{i \rightarrow \infty} P_{s_i} \nabla f(w + h) = \nabla f(w + h) \quad \forall h \in H_0\right\} \\ &\cap \left\{w \in \mathbb{W} : \lim_{i \rightarrow \infty} P_{s_i} \theta_{nm}(w) = \theta_{nm}(w), \forall n, m \in \mathbb{N}\right\} \\ &\cap \left\{w \in \mathbb{W} : |P_{s_i} (\nabla f(\cdot + e^{-s_i} h))(w) - P_{s_i} (\nabla f(\cdot + e^{-s_i} k))(w)|_{X \otimes H} \leq P_{s_i} \theta_{nm}(w), \right. \\ &\quad \left. \forall h, k \in B_n, |h - k|_H < 1/m, \forall i \in \mathbb{N}\right\} \\ &\cap \left\{w \in \mathbb{W} : \lim_{m \rightarrow \infty} \theta_{nm}(w) = 0, \forall n \in \mathbb{N}\right\} \\ &\cap \left\{w \in \mathbb{W} : P_{s_i} \nabla f(w + h) = P_{s_i} \nabla f(w) + \sum_{k=1}^{\infty} \frac{1}{k!} \nabla^k P_{s_i} \nabla f(w) [h^{\otimes k}], \forall h \in H, \forall i \in \mathbb{N}\right\} \\ &\cap \left\{w \in \mathbb{W} : \sum_{k=1}^{\infty} \frac{x^k}{(k+1)!} |\nabla^{k+1} P_{s_i} f(w + h)|_{X \otimes H^{\otimes k+1}} < \infty, \forall h \in H, \forall x \in \mathbb{R}_+, \forall i \in \mathbb{N}\right\} \end{aligned}$$

Observe that we know from [2] that $\left\{w \in \mathbb{W} : \sum_{k=1}^{\infty} \frac{x^n}{(k+1)!} |\nabla^{k+1} P_s f(w)|_{H^{\otimes k+1}} < \infty, \forall x \in \mathbb{R}_+\right\}$ and $\left\{w \in \mathbb{W} : P_{s_i} \nabla f(w+h) = P_{s_i} \nabla f(w) + \sum_{k=1}^{\infty} \frac{1}{k!} \nabla^k P_{s_i} \nabla f(w) [h^{\otimes k}], \forall h \in H\right\}$ are of full measure and H-invariant.

Set $w \in A', i \in \mathbb{N}, h \in H$ and $h' \in H_0$ such that $|h-h'| \leq \frac{1}{m}$ and $n \in \mathbb{N}$ such that $B(h, \frac{1}{m}) \subset B_n$, we have:

$$\begin{aligned}
|P_{s_i} \nabla f(w+h) - \nabla f(w+h)|_{X \otimes H} &\leq |P_{s_i} \nabla f(w+h) - P_{s_i} \nabla f(w+h')|_{X \otimes H} \\
&\quad + |P_{s_i} \nabla f(w+h') - \nabla f(w+h')|_{X \otimes H} \\
&\quad + |\nabla f(w+h') - \nabla f(w+h)|_{X \otimes H} \\
&\leq |P_{s_i}((\nabla f(\cdot + e^{-s_i})h)(w) - P_{s_i}(\nabla f(\cdot + e^{-s_i}h'))(w))|_{X \otimes H} \\
&\quad + |P_{s_i} \nabla f(w+h') - \nabla f(w+h')|_{X \otimes H} \\
&\quad + |\nabla f(w+h') - \nabla f(w+h)|_{X \otimes H} \\
&\leq P_{s_i} \theta_{nm} + |P_{s_i} \nabla f(w+h') - \nabla f(w+h')|_{X \otimes H} + \theta_{nm}
\end{aligned}$$

This proves that:

$$\lim_{i \rightarrow \infty} P_{s_i} \nabla f(w+h) = \nabla f(w+h) \quad \forall h \in H$$

Now observe that for $w \in \mathbb{W}$ such that $(\theta_{n,m}(w))_{m \in \mathbb{N}}$ converges toward 0, $h \mapsto \nabla f(w+h)$ is uniformly continuous on B_n hence bounded. So for $h, k \in B_n$:

$$\begin{aligned}
|f(w+h) - f(w+k)|_X &= \left| \int_0^1 \nabla f(w + \lambda h + (1-\lambda)k)[k-h] d\lambda \right| \\
&\leq \sup_{h' \in B_n} |\nabla f(w+h')|_{X \otimes H} |h-k|_H
\end{aligned}$$

So the hypothesis imply that

$$\lim_{\epsilon \rightarrow 0} \sup_{h, k \in B_n, |h-k|_H \leq \epsilon} |f(w+h) - f(w+k)|_X = 0 \quad a.s.$$

where this supremum is a measurable random variable since $f \in \mathbb{D}_{p,1}(X)$ and we can construct a full measure $A'' \subset \mathbb{W}$ similar to A' where f takes the role of ∇f . We denote $\tilde{A} = A' \cap A''$. We have $\mu(\tilde{A}) = 1$ and for every $w \in \tilde{A}$ and $h \in H$:

$$\begin{aligned}
\lim_{i \rightarrow \infty} P_{s_i} f(w+h) &= f(w+h) \\
\lim_{i \rightarrow \infty} P_{s_i} \nabla f(w+h) &= \nabla f(w+h)
\end{aligned}$$

Set $i, j \geq \max(i_0, i_1, i_2)$, we have:

Now we can prove the differentiability of $h \mapsto f(w+h)$. Set $w \in \tilde{A}$ and $h \in H$, we aim to prove that:

$$\lim_{h' \rightarrow 0} \frac{1}{|h'|_H} |f(w+h+h') - f(w+h) - \nabla f(w+h)[h']|_X = 0$$

Set $h' \in H$, we have:

$$\begin{aligned}
& \frac{1}{|h'|_H} |f(w+h+h') - f(w+h) - \nabla f(w+h)[h']|_X \\
\leq & \frac{1}{|h'|_H} |f(w+h+h') - f(w+h) - (P_{s_i}f(w+h+h') - P_{s_i}f(w+h))|_X \\
& + \frac{1}{|h'|_H} |P_{s_i}f(w+h+h') - P_{s_i}f(w+h) - \nabla P_{s_i}f(w+h)[h']|_X \\
& + \frac{1}{|h'|_H} |\nabla P_{s_i}f(w+h)[h'] - \nabla f(w+h)[h']|_X
\end{aligned}$$

We denote these three terms $A_{h'}$, $B_{h'}$ and $C_{h'}$ and we deal with each one of them separately.

$$\begin{aligned}
C_{h'} & \leq |\nabla P_{s_i}f(w+h) - \nabla f(w+h)|_{X \otimes H} \\
& \leq |e^{-s_i} P_{s_i} \nabla f(w+h) - \nabla f(w+h)|_{X \otimes H} \\
& \rightarrow 0
\end{aligned}$$

now the second term:

$$\begin{aligned}
B_{h'} & = \frac{1}{|h'|_H} \left| \sum_{k=2}^{\infty} \frac{1}{k!} \nabla^k P_{s_i}f(w+h) [h'^{\otimes k}] \right| \\
& \leq \frac{1}{|h'|_H} \sum_{k=2}^{\infty} \frac{1}{k!} |\nabla^k P_{s_i}f(w+h)|_{X \otimes H^{\otimes k}} |h'|_H^k \\
& = \sum_{k=1}^{\infty} \frac{1}{(k+1)!} |\nabla^{k+1} P_{s_i}f(w+h)|_{X \otimes H^{\otimes k+1}} |h'|_H^k \\
& \rightarrow 0
\end{aligned}$$

since $\sum_{k=1}^{\infty} \frac{1}{(k+1)!} |\nabla^{k+1} P_{s_i}f(w+h)|_{X \otimes H^{\otimes k+1}} < \infty$.

$$\begin{aligned}
& A_{h'} \\
= & \lim_{j \rightarrow \infty} \frac{1}{|h'|_H} |P_{s_j}f(w+h+h') - P_{s_j}f(w+h) - (P_{s_i}f(w+h+h') - P_{s_i}f(w+h))|_X
\end{aligned}$$

We have

$$\begin{aligned}
& |P_{s_j}f(w+h+h') - P_{s_j}f(w+h) - (P_{s_i}f(w+h+h') - P_{s_i}f(w+h))|_X \\
\leq & \sup_{\lambda \in [0,1]} |\nabla P_{s_j}f(w+h+\lambda h') - \nabla P_{s_i}f(w+\lambda h+\lambda h')|_{X \otimes H} |h'|_H
\end{aligned}$$

Now set $\lambda \in [0, 1]$, $\epsilon > 0$. We can assume that $|h - h'|_H \leq \frac{1}{m}$, set $n \in \mathbb{N}$ such that $B(h, \frac{1}{m}) \subset B_n$. We have:

$$\begin{aligned}
& \left| \nabla P_{s_j} f(w + h + \lambda h') - \nabla P_{s_i} f(w + h + \lambda h') \right|_{X \otimes H} \\
\leq & \left| \nabla P_{s_j} f(w + h + \lambda h') - \nabla P_{s_j} f(w + h) \right|_{X \otimes H} \\
& + \left| \nabla P_{s_j} f(w + h) - \nabla P_{s_i} f(w + h) \right|_{X \otimes H} \\
& + \left| \nabla P_{s_i} f(w + h) - \nabla P_{s_i} f(w + \lambda h + (1 - \lambda)h') \right|_{X \otimes H} \\
\leq & \left| e^{-s_j} P_{s_j} (\nabla f(\cdot + e^{-s_j}(h + \lambda h')))(w) - e^{-s_j} P_{s_j} (\nabla f(\cdot + e^{-s_j}h))(w) \right|_{X \otimes H} \\
& + \left| \nabla P_{s_j} f(w + h) - \nabla P_{s_i} f(w + h) \right|_{X \otimes H} \\
& + \left| e^{-s_i} P_{s_i} (\nabla f(\cdot + e^{-s_i}h))(w) - e^{-s_i} P_{s_i} (\nabla f(\cdot + e^{-s_i}(h + \lambda h')))(w) \right|_{X \otimes H} \\
\leq & P_{s_i} \theta_{nm}(w + h) + P_{s_j} \theta_{nm}(w + h) + \left| \nabla P_{s_j} f(w + h) - \nabla P_{s_i} f(w + h) \right|_{X \otimes H}
\end{aligned}$$

which is smaller than ϵ for i and j large enough. It ensures that $A_{h'}$ tends toward 0 when h' converges toward 0, which concludes the proof. \square

4. Extension to higher order derivatives

Corollary 1. *Assume that $f : \mathbb{W} \rightarrow X$ is in $\mathbb{D}_{p,r}(X)$ for some $p > 1$. Assume that $h \mapsto \nabla^k f(w + h)$ is μ -a.s. uniformly continuous on every B_n .*

Then f is $H - C^r$ and its H -derivatives up to order n are equal to its weak derivatives of the same order.

Proof: We prove this with a recurrence over n . The case $r = 1$ is theorem 1. Now set $r \leq 2$ and assume that the result is proven for every integer up to $k-1$. Set $n \in \mathbb{N}$ and A a measurable subset of \mathbb{W} such that $\mu(A) = 1$ and for every $w \in A$ $h \mapsto \nabla^r f(w + h)$ is uniformly continuous on B_n . Set $w \in A$, B_n being closed, $h \mapsto \nabla^r f(w + h)$ is bounded on B_n . Consequently, $h \mapsto \nabla^{r-1} f(w + h)$ is lipschitz on B_n and so is uniformly continuous on B_n . The recurrence hypothesis ensures that f is $H - C^{r-1}$ and that its H derivatives up to order $r-1$ are equals to its weak derivatives of the same order.

Applying theorem 1 to $\nabla^{r-1} f$, we get that it is $H - C^1$ and that its H -derivative is $\nabla^r f$, which conclude the proof. \square

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